

## Addendum: Ergodic and Statistical Properties of Piecewise Linear Hyperbolic Automorphisms of the 2-Torus

N. I. Chernov<sup>1</sup>

Received October 20, 1992

When ref. 1 was in press, L. S. Young pointed out a mistake in the proof of a lemma in it to the author. The present note is devoted to the correction of that mistake. We use the notations and the terminology of the original paper.<sup>(1)</sup>

The mistake is in the proof of Lemma 5.5. A counterexample to the proof has been constructed by L. S. Young. The idea is shown in Fig. 1. Here an irregular part of the intersection  $T^p Q \cap Q$ , denoted by  $I_p$ , contains a regular part of  $T^q Q \cap Q$  for some  $q > p > 0$ . If this happens, then we may find a point  $x \in A \cap T^p A \cap T^q A$  such that  $x \in \mathfrak{R}(T^q A \cap A)$  and  $T^{-p}x \in \mathfrak{R}(T^q A \cap A)$ , but  $x \notin \mathfrak{R}(T^p A \cap A)$ . This counterexample destroys the convolution law (5.2) in the proof of Lemma 5.5 of ref. 1 and thus makes the proof incorrect.

Here we present a new version of Lemma 5.5 and prove it. The new version is slightly weaker than the original Lemma 5.5, and it is necessary to modify the proof of the main Theorem 5.1 of ref. 1 as well. We provide the necessary modifications after establishing the new version of Lemma 5.5.

We start our considerations by turning back to the counterexample shown in Fig. 1. The idea is to redefine the parallelogram  $A$  by removing certain parts from the maximal parallelogram  $A(Q)$ .<sup>(1)</sup> In particular, we remove all the points of  $A \cap T^{-p}I_p$  from  $A$ . Precisely, if an irregular part  $I_p$  of  $T^p Q \cap Q$  with a  $p > 0$  can contain a regular part of  $T^p Q \cap Q$  for a  $q > p$ , i.e., if  $I_p$  touches both  $s$ -sides of  $Q$ , then we remove the subset

---

<sup>1</sup> CDSNS, School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332.

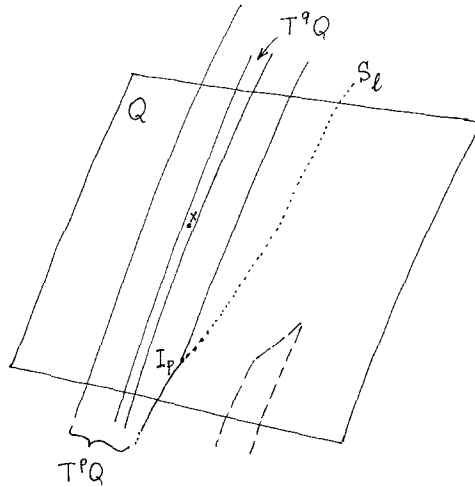


Fig. 1. The counterexample—a quadrilateral  $Q$  intersected by its images  $T^p Q$  and  $T^q Q$ .

$A \cap T^{-p} I_p$  from  $A$ . However, if an irregular part of  $T^q Q \cap Q$  does not touch either of the  $s$ -sides of  $Q$  (e.g., see the dashed contour on Fig. 1), we do not remove its preimage from  $A$ . After removing the specified parts from  $A$  for all  $p > 0$  we get a new parallelogram. We call it the  $s$ -reduced parallelogram and denote it  $A^s$ . Its  $s$ -gaps are wider than those of  $A = A(Q)$ . Likewise, one can define the  $u$ -reduced parallelogram  $A^u$  by applying the same procedure with  $p < 0$ . We work with  $A^s$  only.

First, observe that the counterexample shown in Fig. 1 does not work for the  $s$ -reduced parallelogram  $A^s$ . Therefore, the convolution law (5.2) becomes true if we substitute  $A^s$  for  $A$  in Lemma 5.5. However, the rest of the proof of the lemma is not automatically transferred from  $A$  to  $A^s$ . To complete the proof of the lemma, we first establish certain properties of  $A^s$  similar to the properties of maximal parallelograms obtained in ref. 1. As a by-product of the considerations we also establish the existence of an  $A^s$  with  $m(A^s) > 0$ .

First, note that all the  $s$ -sides of the  $s$ -gaps of  $A^s$  are the preimages of the  $s$ -sides of  $Q$ . The following is an analogue of Lemma 4.2 in ref. 1.

**Lemma 4.2'.** Each irregular part of  $T^p Q \cap Q$ ,  $p > 0$ , contains an irregular part of  $T^p A^s \cap A^s$  (which may be empty). Each regular part of  $T^p Q \cap Q$ ,  $p > 0$ , contains a regular part of  $T^p A^s \cap A^s$ , which may be empty as well.

The last possibility in the lemma, that of emptiness of the regular part of  $T^p A^s \cap A^s$ , is new compared to Lemma 4.2 in ref. 1. The proof of

Lemma 4.2' is rather straightforward. It consists in detailed analysis of all the possible intersections  $T^k Q \cap Q$ . The pre-Markov property of the boundary  $\partial Q$  is heavily used.<sup>(1)</sup> We leave the proof to the reader.

Lemma 4.2' enables us to extend Proposition 5.2 to the  $s$ -reduced parallelogram  $A^s$ , as follows.

**Proposition 5.2'.**  $m(\mathfrak{T}(T^k A^s \cap A^s)) \leq C'_2 (\lambda'_2)^k$  for any  $k > 0$  with some  $C'_2 > 0$  and  $\lambda'_2 \in (0, 1)$  determined by the map  $T$ .

Next, despite the fact that the  $s$ -gaps of  $A^s$  are wider than those of  $A$ , an analogue of Lemma B.1 of ref. 1 still holds for  $A^s$ .

**Lemma B.1'.** The union of all the  $s$ -gaps in  $A^s$  whose widths are less than  $\varepsilon$  has measure less than  $C' \varepsilon^{a'}$ , where  $a' > 0$  is a constant determined by  $T$  and  $C' > 0$  depends on  $A^s$ .

*Proof.* Every  $s$ -gap  $H'$  in  $A^s$  contains a curve  $S_{-l}$  inside with some  $l > 0$ . (If there are more than one curve  $S_{-l}$ , we take the one with the smallest value of  $l$ .) As is shown in the proof of Lemma B.1 of ref. 1, the widths of  $s$ -gaps with curves  $S_{-l}$  for  $l < -C_2 \ln \varepsilon$  are greater than  $\varepsilon$ . Consider a gap  $H'$  with a curve  $S_{-l}$  for an  $l \geq -C_2 \ln \varepsilon$ . It is the union of the corresponding gap  $H$  of the maximal parallelogram  $A = A(Q)$  and two additional strips adjacent to both  $s$ -sides of  $H$  specified in the construction of  $A^s$ . Precisely, for some  $l' > 0$  the image  $T^{l+l'} H'$  contains an irregular part  $I'$  of  $T^{l+l'} Q \cap Q$  which touches both  $s$ -sides of  $Q$ , and then its preimage  $T^{-(l+l')} I'$  is added to  $H$  in the construction of  $A^s$ , thus producing the wider gap  $H'$ . The total measure of all those parts  $I'$  for all  $l' > 0$  and  $l \geq -C_2 \ln \varepsilon$  is less than  $\text{const} \cdot (\lambda)^{-C_2 \ln \varepsilon}$  due to Proposition 5.2 of ref. 1. Hence the lemma.

**Corollary.** The set of points inside the  $s$ -gaps of  $A^s$  located in the  $\varepsilon$ -neighborhood of the  $s$ -sides of those gaps has measure  $\leq C'' \varepsilon^{a''}$  for some  $C'' > 0$  depending on  $A^s$ .

Now we are ready to extend the main technical Proposition 5.3 of ref. 1 to the  $s$ -reduced parallelogram  $A^s$ . Let  $B$  denote an arbitrary maximal parallelogram or let  $B = A^s$ . Define, as in ref. 1, the function  $k_{B, A^s}^+(x)$  for  $x \in B$  as the minimum positive  $k$  such that  $T^k x \in \mathfrak{R}(T^k B \cap A^s)$ , and let  $p_{B, A^s}^+(k) = m\{x \in B: k_{B, A^s}^+(x) = k\} / m(B)$ .

**Proposition 5.3'.** If  $B$  is a maximal parallelogram or if  $B = A^s$ , then one has  $p_{B, A^s}^+(k) \leq C'_3 (\lambda'_3)^k / m(B)$  for every  $k > 0$ . Here  $C'_3 > 0$  and  $\lambda'_3 \in (0, 1)$  are determined by the map  $T$  and the  $s$ -reduced parallelogram  $A^s$ .

*Proof.* Note that the  $s$ -reduced parallelogram  $A^s$  plays the same role here as the maximal parallelogram  $B$  in Proposition 5.3 of ref. 1. The proof

of the latter was given in Appendix B in ref. 1. Its first step—Lemma B.1—is extended to the reduced parallelogram  $A^s$  by Lemma B.1'. The following Lemmas B.2 and B.3 of ref. 1 did not involve any parallelograms. Lemmas B.4 and B.5 were based on Lemma B.1, so they are transferable to the present case automatically. The rest of the proof of Proposition 5.3 applies to Proposition 5.3' word for word.

Now we state and prove a weaker version of Lemma 5.5 of ref. 1.

**Lemma 5.5'.** For every positive integer  $k$  one has  $m(\mathfrak{R}(T^k A^s \cap A^s)) \geq m(A^s)^2 (1 - C'_5(\lambda'_5)^k)$  with some constants  $C'_5 > 0$  and  $\lambda'_5 \in (0, 1)$  determined by  $A^s$ .

*Proof.* As already stated, the convolution law (5.2) of ref. 1 now holds. Propositions 5.2' and 5.3' allow one to accomplish the proof of Lemma 5.5' in the same way as that of Lemma 5.5.<sup>(1)</sup>

Now we adjust the rest of the proof of property G3 (see the end of Section 5 in ref. 1) to the  $s$ -reduced parallelogram  $A^s$  instead of the maximal parallelogram  $A$ . As in ref. 1, take two parallelograms  $A_i, A_j \in G_n$ . Then consider the function  $k^+_{A_i, A^s}(x)$  on  $A_i$  defined above. It generates a partition of  $A_i$  into  $s$ -inscribed subparallelograms  $A_{i,1}, A_{i,2}, \dots$  on which that function is constant. For each  $p \geq 1$ , denote by  $k^+_{i,p}$  the value taken by that function on  $A_{i,p}$ . The image  $T^{k^+_{i,p}} A_{i,p}$  is a parallelogram  $u$ -inscribed in  $A^s$  for each  $p \geq 1$ . Note that the same property for the maximal parallelogram  $A$  instead of  $A^s$ , stated in ref. 1, Section 5, was wrong again due to the counterexample shown in Fig. 1. I do not explain this in detail.

On the contrary, the partition of  $A_j$  into  $A_{j,1}, A_{j,2}, \dots$  is different from that in ref. 1. Its definition here is inductive. First, for any maximal parallelogram  $B$ , denote by  $\mathfrak{R}^s(T^{-l}B \cap A)$  the union of the parallelograms in  $\mathfrak{R}(T^{-l}B \cap A)$  whose distance from the  $s$ -reduced parallelogram  $A^s$  is less than  $\lambda^l$ . (Recall that  $\lambda$  is the smallest eigenvalue of the matrix of the map  $T$ ; see Section 2 of ref. 1.) We find

$$k^-_{j,1} = \min\{k \geq n/10: \mathfrak{R}^s(T^{-k}A_j \cap A) \neq \emptyset\}$$

and set

$$A_{j,1} = T^{k^-_{j,1}}(\mathfrak{R}^s(T^{-k^-_{j,1}}A_j \cap A))$$

and  $A'_{j,1} = A_j \setminus A_{j,1}$ . Then, for each  $q \geq 1$ , we find

$$k^-_{j,q+1} = \min\{k > k^-_{j,q}: \mathfrak{R}^s(T^{-k}A'_{j,q} \cap A) \neq \emptyset\}$$

and set

$$A_{j,q+1} = T^{k^-_{j,q+1}}(\mathfrak{R}^s(T^{-k^-_{j,q+1}}A'_{j,q} \cap A))$$

and  $A'_{j,q+1} = A_j \setminus (A_{j,1} \cup \dots \cup A_{j,q})$ . Certainly, we get a partition  $A_j = \bigcup A_{j,q}$  into subparallelograms  $u$ -inscribed in  $A_j$  such that  $T^{-k^-_{j,q}}A_{j,q}$  are

parallelograms  $s$ -inscribed in  $A$ . Note that we used the maximal parallelogram  $A = A(Q)$  instead of  $A^s$ .

**Proposition 5.3''.** For any  $j$  and  $n \geq 1$  one has

$$m\left(\bigcup_{q: k_{j,q}^- \geq n} A_{j,q}\right) \leq C_3''(\lambda_3'')^n$$

where  $C_3'' > 0$  and  $\lambda_3'' \in (0, 1)$  are determined by the parallelogram  $A$  alone.

This is a slightly stronger version of Proposition 5.3 of ref. 1. A careful examination of the proof of the latter<sup>(1)</sup> shows that actually it applies to Proposition 5.3'' as well, except for one point discussed below.

The point here is our requirement that the distance of the parallelograms  $T^{-k_{j,q}^-}A_{j,q}$  from the  $s$ -reduced parallelogram  $A^s$  is less than  $\lambda^{k_{j,q}^-}$ . To adjust the proof of Proposition 5.3 to this extra requirement, we strengthen Lemma B.3 of ref. 1.

**Lemma B.3'.** For every  $\rho > 0$  and every parallelogram  $A'$  such that  $m(A') > 0$ , there are  $n_0 \geq 0$  and  $\alpha_0 > 0$  such that  $m_1(\gamma_n''(A')) \geq \alpha_0$  for every LUM  $\gamma''$  of length  $\rho$  and every  $n \geq n_0$ .

The notation  $\gamma_n''(A')$ , unlike  $\gamma_n''(Q)$  used in Lemma B.3 of ref. 1, stands here for the union of all the subintervals in  $\gamma''$  whose images under  $T^n$  are LUMs inscribed in  $Q(A')$  and located within the distance  $\leq \lambda^n$  from  $A'$ .

The proof of Lemma B.3 of ref. 1 can be easily adjusted to Lemma B.3' due to the following observation. The mixing property of  $T$  implies that  $m(T^n B \cap A') > m(B) m(A')/2$  for all  $n > n_0(A', B)$  (see ref. 1 for notations). Whenever a quadrilateral in  $T^n Q(B) \cap Q(A')$  intersects also  $A'$ , the part of  $T^n \gamma''$  inside that quadrilateral is located within the distance  $\lambda^n$  from  $A'$  due to the hyperbolicity of  $T$ . This completes the proofs of Lemma B.3' and Proposition 5.3''.

We now set

$$B_{i,p} = T^{k_{i,p}^+} A_{i,p} \quad \text{and} \quad B_{j,q} = (T^{-k_{j,q}^-} A_{j,q}) \cap A^s$$

Obviously,

$$m(T^n A_i \cap A_j) = \sum_{p,q} m(T^n A_{i,p} \cap A_{j,q}) \geq \sum_{p,q} m(T^{n_{pq}} B_{i,p} \cap B_{j,q})$$

where  $n_{pq} = n - k_{i,p}^+ - k_{j,q}^-$ . Since  $B_{i,p}$  are  $u$ -inscribed in  $A^s$  and  $B_{j,q}$  are  $s$ -inscribed in  $A^s$  we can apply formula (4.1) of ref. 1 and then use Lemma 5.5' to get an inequality similar to (5.5) in ref. 1:

$$m(T^{n_{pq}} B_{i,p} \cap B_{j,q}) \geq m(B_{i,p}) m(B_{j,q}) [1 - C_5'(\lambda_5')^{n_{pq}}]$$

provided  $n_{pq} > 0$ .

As in ref. 1, we have to sacrifice all the parallelograms  $B_{i,p}$  ( $B_{j,q}$ ) such that  $k_{i,p}^+ > n/3$  ( $k_{j,q}^- > n/3$ ). Their union has measure less than  $C_3 \lambda_3^{n/3}$  due to Propositions 5.3' and 5.3'', where  $C_3 = 2 \max\{C_3', C_3''\}$  and  $\lambda_3 = \max\{\lambda_3', \lambda_3''\}$ . Summing up the above inequality over  $p, q$  gives

$$m(T^n A_i \cap A_j) \geq [m(A_i) - C_3 \lambda_3^{n/3}][m(A_j') - C_3 \lambda_3^{n/3}][1 - C_3' (\lambda_5')^{n/3}]$$

where  $A_j' = \bigcup T^{k_{j,q}^-} B_{j,q}$  is a subparallelogram in  $A_j$ . The complement  $A_j \setminus A_j'$  consists of the subsets  $T^{k_{j,q}^-}(T^{-k_{j,q}^-} A_{j,q} \setminus A^s)$ ,  $q \geq 1$ . For each  $q \geq 1$  the set  $T^{-k_{j,q}^-} A_{j,q} \setminus A^s$  is located in the union of the  $s$ -gaps of  $A^s$  and, moreover, in the  $(\lambda^{k_{j,q}^-})$ -neighborhood of the  $s$ -sides of those gaps. Applying the corollary after the Lemma B.1' and then summing up over all  $k_{j,q}^- \geq n/10$  gives the estimate  $m(A_j \setminus A_j') \leq \text{const} \cdot \lambda^{na'/10}$ . Recall that  $m(A_i) \geq \text{const} \cdot \lambda_4^{2n}$  for every parallelogram  $A_i \in G$ ,<sup>(1)</sup> and then we can complete the proof of property G3 by setting  $\lambda_4 = \max\{\lambda^{a'/30}, \lambda_3^{1/9}\}$  and  $\lambda_1 = \max\{\lambda_4, (\lambda_5')^{1/3}\}$ .

Finally, we establish the existence of an  $s$ -reduced parallelogram  $A^s$  with a positive measure. We take a pre-Markov partition  $\xi_0 = \xi_0(\varepsilon)$ ,<sup>(1)</sup> with a sufficiently small  $\varepsilon$ . Denote by  $Q_i$ ,  $1 \leq i \leq I$ , all the quadrilaterals in  $\xi_0$  and  $A_i = A(Q_i)$ . Lemma 5.4 of ref. 1 ensures that  $m(\bigcup A_i) \geq 1 - \text{const} \cdot \sqrt{\varepsilon}$ .

We now estimate the measure of the set  $\bigcup (A_i \setminus A_i^s)$ , where  $A_i^s$  is the  $s$ -reduced parallelogram constructed out of the maximal parallelogram  $A_i$ . This set can be covered with the preimages  $T^{-p} I_p$  of the irregular parts  $I_p$  of  $T^p Q_i \cap Q_i$ ,  $p \geq 1$ ,  $1 \leq i \leq I$ , involved in our construction of  $s$ -reduced parallelograms. Each such part  $I_p$  touches a singularity curve  $S_l$  for some  $l$ ,  $1 \leq l \leq p$ , inside the corresponding quadrilateral  $Q_i$ , like the one shown in Fig. 1. Fix a pair of integers  $p \geq 1$  and  $l \in [1, p]$  and collect all the pertinent parts  $I_p$  in all the quadrilaterals  $Q_i$ ,  $1 \leq i \leq I$ . Each of those parts has measure  $\leq \text{const} \cdot \varepsilon^2 \lambda^p$ , and their number is less than  $\text{const} \cdot A_1^l / \varepsilon$  with some  $A_1 > 1$ . On the other hand, their preimages  $T^{-(l-1)} I_p$  are disjoint and located within the  $(\varepsilon \lambda^l)$ -neighborhood of the singularity set  $S_+$ . Therefore, the total measure of all those parts  $I_p$  with the values  $p$  and  $l$  fixed above is less than

$$\text{const} \cdot \min\{\varepsilon \lambda^p A_1^l, \varepsilon \lambda^l\}$$

Adding these quantities for all  $p \geq 1$  and  $1 \leq l \leq p$  gives the bound

$$m\left(\bigcup (A_i \setminus A_i^s)\right) \leq \text{const} \cdot \varepsilon$$

Hence,  $m(\bigcup A_i^s) \geq 1 - \text{const} \cdot \sqrt{\varepsilon}$  and the proof is completed.

Assessing the problem with the original proof of Lemma 5.5 of ref. 1 illustrated in Fig. 1 and our way of solving it, we have to admit that our

general idea stated in Section 4 of ref. 1 was not quite accurate. The crude approximations to the Markov partitions which we call the Markov sieves are good for establishing a subexponential decay of correlations. However, the exponential decay requires more than that. Namely, our  $s$ -reduced parallelogram here is trickier and more sophisticated than just a maximal parallelogram used in ref. 1. In a sense, it is almost an element of a Markov partition. Nonetheless, our technique still does not involve the Markov partitions, which are fairly difficult to construct and to use in dynamical systems with singularities.

## ACKNOWLEDGMENT

I thank L. S. Young for pointing out the mistake in ref. 1 and for illuminating discussions.

## REFERENCES

1. N. I. Chernov, Ergodic and statistical properties of piecewise linear hyperbolic automorphisms of the 2-torus, *J. Stat. Phys.* **69**:111–134 (1992).